

On C_2 -cofiniteness of parafermion vertex operator algebras

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Abstract

It is proved that the regularity of parafermion vertex operator algebras associated to integrable highest weight modules for affine Kac-Moody algebra $A_1^{(1)}$ implies the C_2 -cofiniteness of parafermion vertex operator algebras associated to integrable highest weight modules for any affine Kac-Moody algebra. In particular, the parafermion vertex operator algebra associated to an integrable highest weight module of small level for any affine Kac-Moody algebra is C_2 -cofinite and has only finitely many irreducible modules. Also, the parafermion vertex operator algebras with level 1 are determined explicitly.

1 Introduction

This paper is devoted to the study of the C_2 -cofiniteness of parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated to integrable highest weight module of level k for affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, where \mathfrak{g} is a finite dimensional simple Lie algebra. It is established that the regularity of $K(sl_2, k)$ for all k implies the C_2 -cofiniteness of $K(\mathfrak{g}, k)$ for all k . In particular, the C_2 -cofiniteness of $K(\mathfrak{g}, k)$ is obtained if \mathfrak{g} is of ADE type and $k \leq 6$, \mathfrak{g} is of type G_2 and $k \leq 2$, and \mathfrak{g} is of type B_l, C_l, F_4 and $k \leq 3$. The structure and representation theory of $K(\mathfrak{g}, 1)$ is analyzed in details. It turns out that $K(\mathfrak{g}, 1)$ for \mathfrak{g} being of type B_l, C_l, F_4, G_2 is isomorphic to the parafermion vertex operator algebra of type A with level $k = 2$ or 3 .

The origin of the parafermion vertex operator algebras is the Z -algebras developed in [22], [23], [24] in the construction of integrable highest weight modules for affine Kac-Moody algebras. It was investigated in [28] how the Z -algebras and Z -operators lead to a new conformal field theory – parafermion conformal field theory. This further stimulated the development of the theory of generalized vertex operator algebras [6] which, in turns, provides a framework for the parafermion conformal field theory. It is well-known that the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ contains a Heisenberg algebra. As a result, the vertex operator algebra $\mathcal{L}(k, 0)$ associated to the highest weight module of level k for $\widehat{\mathfrak{g}}$ contains the Heisenberg vertex operator algebra as a subalgebra. The parafermion vertex operator

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algebra $K(\mathfrak{g}, k)$ is the commutant [14], [21] of the Heisenberg vertex operator algebra in $\mathcal{L}(k, 0)$, and is a special kind of coset construction [17].

Some aspects of both the structure and representation theory of parafermion vertex operator algebras have been studied in [3]-[5],[11]. In particular, a set of generators for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is obtained. It is also shown that $K(\mathfrak{g}, k)$ is generated by $K(sl_2, k_\alpha)$ for positive roots α , where $k_\alpha \in \{k, 2k, 3k\}$ is determined by the squared length of α [11]. This suggests that the structure and representation theory for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ can be understood by using the structure and representation theory of $K(sl_2, k_\alpha)$ (see [11] for details).

The rationality [29],[8] on the semisimplicity of the admissible module category and C_2 -cofiniteness [29] on the cofiniteness of certain subspace of the vertex operator algebra are perhaps two most important concepts in the representation theory of vertex operator algebras. It was proved in [26] and [1] that the rationality together with C_2 -cofiniteness is equivalent to the regularity [7] which says that any weak module is a direct sum of irreducible ordinary modules. Many well known vertex operator algebras such as vertex operator algebras associated to positive definite even lattices, integrable highest weight modules for the affine Kac-Moody algebras, highest weight modules associated to the minimal series for the Virasoro algebras are regular [7]. It is natural to expect the rationality and C_2 -cofiniteness of the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ according to the general principle in the coset theory: the commutant of a regular vertex operator subalgebra in a regular vertex operator algebra is again regular. On the surface, $K(\mathfrak{g}, k)$ is the commutant of the Heisenberg vertex operator algebra which is neither rational nor C_2 -cofinite in regular vertex operator algebra $\mathcal{L}(k, 0)$. But the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ can also be realized as the commutant of lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ in $\mathcal{L}(k, 0)$, where Q_L is the positive definite even lattice generated by the long roots of \mathfrak{g} (for example, see [20],[6],[4]). There is no doubt that the representation theory of $K(\mathfrak{g}, k)$ would be rich and interesting.

As we have already mentioned that we hope some important properties such as C_2 -cofiniteness and rationality for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ can be obtained by studying the simplest parafermion vertex operator algebra associated to $\widehat{sl_2}$. In this paper, we succeed in proving that the C_2 -cofiniteness for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is completely determined by the regularity of $K(sl_2, k)$. It was shown in [4] that $K(sl_2, k)$ is rational and C_2 -cofinite for $k \leq 6$, thus the C_2 -cofiniteness for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ of small level k follows. The main idea in proving this result is to use the generating result given in [11], the property of the hermitian operator in the unitary representation theory for the Virasoro algebra and the approach in [26] to the maximal weak module in a completion of an ordinary module.

Although the generators of general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ are determined in [11], it is far from over to understand the structure theory for $K(\mathfrak{g}, k)$ completely. The main challenging is to figure out how the generators act on each other or how the generators are related. As an experiment, we study the simplest case $K(\mathfrak{g}, k)$ with $k = 1$ in great details. It is clear that $K(\mathfrak{g}, 1) = \mathbb{C}$ if \mathfrak{g} is of ADE type. So the main discussion is put on the non-simply laced simple Lie algebras, i.e., those of type B_l , C_l , F_4 , and G_2 . A critical observation in the analysis of $K(\mathfrak{g}, 1)$ for \mathfrak{g} being non-simply

laced simple Lie algebra is Lemma 4.2 which tells us the relation between the generators associated to the short roots. The result can be summarized as follows: $K(B_l, 1)$ is isomorphic to $K(A_1, 2)$ which is the Virasoro vertex operator algebra associated to the irreducible highest weight module for the Virasoro algebra with central charge $\frac{1}{2}$, $K(C_l, 1)$ is isomorphic to $K(A_{l-1}, 2)$, $K(F_4, 1)$ is isomorphic to $K(A_2, 2)$ and $K(G_2, 1)$ is isomorphic to $K(A_1, 3)$.

The paper is organized as follows. In Section 2, we fix the setting, recall the definition of parafermion vertex operator algebra $K(\mathfrak{g}, k)$, and present some results on parafermion vertex operator algebra $K(\mathfrak{g}, k)$ from [3], [4], [5], [11]. In Section 3, we prove the main result in this paper. That is, the regularity of $K(sl_2, k)$ implies the C_2 -cofiniteness of $K(\mathfrak{g}, k')$, where k' depends on k and the Lie algebra \mathfrak{g} . The proof involves with the positive definite hermitian form and the nondegenerate symmetric invariant bilinear form on $K(\mathfrak{g}, k')$. While the hermitian form allows us to study certain hermitian operators on a completion of $K(\mathfrak{g}, k')$, the bilinear form gives us the identification of $K(\mathfrak{g}, k')$ with its contragredient module. The semisimplicity of the hermitian operator associated to any root of \mathfrak{g} forces the maximal weak $K(\mathfrak{g}, k')$ -module in the completion of $K(\mathfrak{g}, k')$ to be $K(\mathfrak{g}, k')$ itself. The C_2 -cofiniteness of $K(\mathfrak{g}, k')$ is immediate by a result in [26]. In Section 4, we discuss the structure of $K(\mathfrak{g}, 1)$ as a starting point to the follow-up work on the representation theory of parafermion vertex operator algebras.

We expect the reader to be familiar with the elementary theory of vertex operator algebras as found, for example, in [13] and [21].

2 Parafermion vertex operator algebras $K(\mathfrak{g}, k)$

We are working in the setting of [11]. Fix a finite dimensional simple Lie algebra \mathfrak{g} with a Cartan subalgebra \mathfrak{h} . We use Δ and Q to denote the corresponding root system and root lattice, respectively. We also fix an invariant symmetric nondegenerate bilinear form \langle, \rangle on \mathfrak{g} so that $\langle \alpha, \alpha \rangle = 2$ if α is a long root, where we have identified \mathfrak{h} with \mathfrak{h}^* via \langle, \rangle . As in [18], for $\alpha \in \mathfrak{h}^*$, we denote its image in \mathfrak{h} by t_α , that is, $\alpha(h) = \langle t_\alpha, h \rangle$ for any $h \in \mathfrak{h}$. Let $\{\alpha_1, \dots, \alpha_l\}$ be the simple roots and denote the highest root by θ .

Let \mathfrak{g}_α denote the root space associated to the root $\alpha \in \Delta$. For $\alpha \in \Delta_+$, we fix $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} t_\alpha \in \mathfrak{h}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$, $[h_\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$. That is, $\mathfrak{g}^\alpha = \mathbb{C}x_\alpha + \mathbb{C}h_\alpha + \mathbb{C}x_{-\alpha}$ is isomorphic to sl_2 by sending x_α to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $x_{-\alpha}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and h_α to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\langle h_\alpha, h_\alpha \rangle = \frac{4}{\langle \alpha, \alpha \rangle}$ and $\langle x_\alpha, x_{-\alpha} \rangle = \frac{2}{\langle \alpha, \alpha \rangle}$ for all $\alpha \in \Delta$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the corresponding affine Lie algebra. Fix a positive integer k and let

$$V(k, 0) = V_{\widehat{\mathfrak{g}}}(k, 0) = Ind_{\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}$$

be the induced module where $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as 0 and K acts as k on $\mathbb{1} = 1$. Then $V(k, 0)$ is a vertex operator algebra generated by $a(-1)\mathbb{1}$ for $a \in \mathfrak{g}$ such that

$$Y(a(-1)\mathbb{1}, z) = a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$

where $a(n) = a \otimes t^n$, with the vacuum vector $\mathbb{1}$ and the Virasoro vector

$$\omega_{\text{aff}} = \frac{1}{2(k + h^\vee)} \left(\sum_{i=1}^l h_i(-1)h_i(-1)\mathbb{1} + \sum_{\alpha \in \Delta} \frac{\langle \alpha, \alpha \rangle}{2} x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1} \right)$$

of central charge $\frac{k \dim \mathfrak{g}}{k + h^\vee}$ (e.g. [14], [20], [21, Section 6.2]), where h^\vee is the dual Coxeter number of \mathfrak{g} and $\{h_i | i = 1, \dots, l\}$ is an orthonormal basis of \mathfrak{h} .

Let $M(k)$ be the vertex operator subalgebra of $V(k, 0)$ generated by $h(-1)\mathbb{1}$ for $h \in \mathfrak{h}$ with the Virasoro element

$$\omega_{\mathfrak{h}} = \frac{1}{2k} \sum_{i=1}^l h_i(-1)h_i(-1)\mathbb{1}$$

of central charge l , where $\{h_1, \dots, h_l\}$ is an orthonormal basis of \mathfrak{h} as before.

As usual we denote the component operators of $Y(u, z)$ for $u \in V$ by u_n for any vertex operator algebra V . That is, $Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$. In the case $V = V(k, 0)$ and $u = a(-1)\mathbb{1}$ for $a \in \mathfrak{g}$, we see that $(a(-1)\mathbb{1})_n = a(n)$. So in the rest of paper, we will use both $a(n)$ and $(a(-1)\mathbb{1})_n$ for $a \in \mathfrak{g}$ and use u_n only for general u without further explanation.

We denote the unique irreducible quotient $\widehat{\mathfrak{g}}$ -module of $V(k, 0)$ by $\mathcal{L}(k, 0)$. Then $\mathcal{L}(k, 0)$ is a simple, rational vertex operator algebra. Moreover, the image of $M(k)$ in $\mathcal{L}(k, 0)$ is isomorphic to $M(k)$ and will be denoted by $M(k)$ again. Set

$$K(\mathfrak{g}, k) = \{v \in \mathcal{L}(k, 0) \mid h(m)v = 0 \text{ for } h \in \mathfrak{h}, m \geq 0\}.$$

Then $K(\mathfrak{g}, k)$ which is the space of highest weight vectors with highest weight 0 for $\widehat{\mathfrak{h}}$ is the commutant of $M(k)$ in $\mathcal{L}(k, 0)$ and is called the parafermion vertex operator algebra associated to the irreducible highest weight module $\mathcal{L}(k, 0)$ for $\widehat{\mathfrak{g}}$. The Virasoro element of $K(\mathfrak{g}, k)$ is given by

$$\omega = \omega_{\text{aff}} - \omega_{\mathfrak{h}},$$

where we still use $\omega_{\text{aff}}, \omega_{\mathfrak{h}}$ to denote their images in $\mathcal{L}(k, 0)$.

Let $\alpha \in \Delta$ and set $k_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} k$. Note that $\widehat{\mathfrak{g}}^\alpha$ is a subalgebra of $\widehat{\mathfrak{g}}$ and $\mathcal{L}(k, 0)$ is an integrable $\widehat{\mathfrak{g}}^\alpha$ -module of level k_α .

For $\alpha \in \Delta$, we set

$$\begin{aligned} \omega_\alpha &= \frac{1}{2k_\alpha(k_\alpha + 2)} (-k_\alpha h_\alpha(-2)\mathbb{1} - h_\alpha(-1)^2\mathbb{1} + 2k_\alpha x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1}) \\ &= \frac{1}{2k_\alpha(k_\alpha + 2)} (-h_\alpha(-1)^2\mathbb{1} + k_\alpha x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1} + k_\alpha x_{-\alpha}(-1)x_\alpha(-1)\mathbb{1}) \end{aligned}$$

$$\begin{aligned} W_\alpha^3 &= k_\alpha^2 h_\alpha(-3)\mathbb{1} + 3k_\alpha h_\alpha(-2)h_\alpha(-1)\mathbb{1} + 2h_\alpha(-1)^3\mathbb{1} - 6k_\alpha h_\alpha(-1)x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1} \\ &\quad + 3k_\alpha^2 x_\alpha(-2)x_{-\alpha}(-1)\mathbb{1} - 3k_\alpha^2 x_\alpha(-1)x_{-\alpha}(-2)\mathbb{1}. \end{aligned}$$

It is easy to see that $\omega_\alpha = \omega_{-\alpha}$ and $W_{-\alpha}^3 = -W_\alpha^3$ for $\alpha \in \Delta$. A straightforward verification shows that

$$\omega = \sum_{\alpha \in \Delta_+} \frac{k(k_\alpha + 2)}{k_\alpha(k + h^\vee)} \omega_\alpha.$$

The following result can be found in [1], [4], [5] and [11].

Theorem 2.1. (1) Vertex operator algebra $K(\mathfrak{g}, k)$ is generated by $\omega_\alpha, W_\alpha^3$ for $\alpha \in \Delta_+$.
(2) $K(sl_2, k)$ is a simple rational and C_2 -cofinite for $k \leq 6$. That is, $K(sl_2, k)$ is regular for such k .
(3) Let P_α be the vertex operator subalgebra of $K(\mathfrak{g}, k)$ generated by $\omega_\alpha, W_\alpha^3$. Then P_α is isomorphic to $K(sl_2, k_\alpha)$ as vertex operator algebras.

3 C_2 -cofiniteness of $K(\mathfrak{g}, k)$

We need to recall some definitions from [29], [7], [8]. A vertex operator algebra V is called C_2 -cofinite if $\dim V/C_2(V) < \infty$, where $C_2(V)$ is a subspace of V spanned by $u_{-2}v$ for $u, v \in V$. V is called regular if any weak module is a direct sum of irreducible ordinary modules. V is called rational if any admissible module is completely reducible. It was proved in [26] and [1] that the rationality together with C_2 -cofiniteness is equivalent to the regularity. The most important property of C_2 -cofinite vertex operator algebra is that such vertex operator algebra is finitely generated and has a PBW type spanning set [15]. Furthermore, C_2 -cofinite vertex operator algebra has only finitely many irreducible admissible modules up to isomorphism and each irreducible admissible module is ordinary [26], [15].

In this section, we prove the main theorem of this paper.

Theorem 3.1. Let q be a positive integer. If $K(sl_2, k)$ is rational and C_2 -cofinite for $k \leq q$, then $K(\mathfrak{g}, k)$ is C_2 -cofinite if \mathfrak{g} is ADE type and $k \leq q$, \mathfrak{g} is type G_2 and $k \leq [q/3]$, and \mathfrak{g} is other type and $k \leq [q/2]$, where $[r]$ denotes the maximal integer less than or equal to r for any real number r .

So the C_2 -cofiniteness for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ follows from the regularity of vertex operator algebra $K(sl_2, k)$ for all k . It is expected that the rationality of general parafermion vertex operator algebra also follows the regularity of $K(sl_2, k)$ for all k . So the study of simplest parafermion vertex operator algebras $K(sl_2, k)$ is crucial in understanding the general parafermion vertex operator algebras.

Combining Theorem 2.1, Theorem 3.1 and [1, Proposition 5.7], we immediately have

Corollary 3.2. $K(\mathfrak{g}, k)$ is C_2 -cofinite if \mathfrak{g} is ADE type and $k \leq 6$, \mathfrak{g} is type G_2 and $k \leq 2$, and \mathfrak{g} is other type and $k \leq 3$. In particular, there are only finitely many irreducible modules up to isomorphism and each irreducible weak module is ordinary for such vertex operator algebra.

In order to prove the theorem, we need some facts on the invariant bilinear form on $K(\mathfrak{g}, k)$ and some related results from [25]-[26]. Note that $K(\mathfrak{g}, k) = \bigoplus_{n \geq 0} K(\mathfrak{g}, k)_n$ with $K(\mathfrak{g}, k)_0 = \mathbb{C}\mathbb{1}$ and $K(\mathfrak{g}, k)_1 = 0$. It follows from [25] that there is a unique nondegenerate symmetric bilinear form [12] $\langle \cdot, \cdot \rangle$ on $K(\mathfrak{g}, k)$ such that

$$\langle \mathbb{1}, \mathbb{1} \rangle = 1$$

$$\langle Y(u, z)v, w \rangle = \langle v, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w \rangle$$

for $u, v, w \in K(\mathfrak{g}, k)$, where $L(n)$ are the component operator of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$. As a result, $K(\mathfrak{g}, k)$ is isomorphic to its contragredient module $K(\mathfrak{g}, k)' = \sum_{n \in \mathbb{Z}} K(\mathfrak{g}, k)_n^*$, where $K(\mathfrak{g}, k)_n^*$ is the dual space of $K(\mathfrak{g}, k)_n$ [12]. Denote by $\overline{K(\mathfrak{g}, k)}$ the direct product of $K(\mathfrak{g}, k)_n$ for $n \in \mathbb{Z}$. Each vector in $\overline{K(\mathfrak{g}, k)}$ can be written as

$$(v^0, v^1, \dots, v^n, \dots),$$

where $v^i \in K(\mathfrak{g}, k)_i$. Then $\overline{K(\mathfrak{g}, k)}$ can be identified with $K(\mathfrak{g}, k)^*$ naturally by using the bilinear form \langle, \rangle .

Recall from [2] the Lie algebra $\widehat{K(\mathfrak{g}, k)} = K(\mathfrak{g}, k) \otimes \mathbb{C}[t^{\pm 1}] / D(K(\mathfrak{g}, k) \otimes \mathbb{C}[t^{\pm 1}])$, where $D = L(-1) \otimes 1 + 1 \otimes t \frac{d}{dt}$. Let $a_{(n)}$ be the image of $a \otimes t^n$ in $\widehat{K(\mathfrak{g}, k)}$ for $a \in K(\mathfrak{g}, k)$ and $n \in \mathbb{Z}$. The Lie bracket in $\widehat{K(\mathfrak{g}, k)}$ is defined as follows:

$$[a_{(m)}, b_{(n)}] = \sum_{i=0}^{m+n-1} (a_i b)_{(m+n-i)}$$

for homogeneous $a, b \in K(\mathfrak{g}, k)$ and $m, n \in \mathbb{Z}$. Then $K(\mathfrak{g}, k)$ is a $\widehat{K(\mathfrak{g}, k)}$ -module such that $a_{(n)}$ acts as a_n and extend the action to $\overline{K(\mathfrak{g}, k)}$ in an obvious way to make $\overline{K(\mathfrak{g}, k)}$ a $\widehat{K(\mathfrak{g}, k)}$ -module [12], [26]. Let $\mathcal{D}(K(\mathfrak{g}, k))$ be the subspace of $\overline{K(\mathfrak{g}, k)}$ consisting of vectors u such that $a_{(n)}u = 0$ for $a \in K(\mathfrak{g}, k)$ and n sufficiently large. Then $\mathcal{D}(K(\mathfrak{g}, k))$ is a weak $K(\mathfrak{g}, k)$ -module and $K(\mathfrak{g}, k)$ is a submodule of $\mathcal{D}(K(\mathfrak{g}, k))$ [26].

We also need to use a positive definite hermitian form on $K(\mathfrak{g}, k)$ in the proof of theorem. It is well known that $\mathcal{L}(k, 0)$ is a unitary representation for the affine Lie algebra $\widehat{\mathfrak{g}}$ [20]. In fact, there is a unique positive definite hermitian form $(,)$ on $\mathcal{L}(k, 0)$ such that

$$\begin{aligned} (\mathbb{1}, \mathbb{1}) &= 1, \\ (h_\alpha(m)u, v) &= (u, h_\alpha(-m)v), \\ (x_\alpha(m)u, v) &= (u, x_{-\alpha}(-m)v) \end{aligned}$$

for $\alpha \in \Delta$ and $u, v \in \mathcal{L}(k, 0)$, $m \in \mathbb{Z}$. Clearly, the restriction of $(,)$ to $K(\mathfrak{g}, k)$ defines a positive definite hermitian form on $K(\mathfrak{g}, k)$.

Set

$$Y(\omega_\alpha, z) = \sum_{n \in \mathbb{Z}} L_\alpha(n)z^{-n-2}$$

for $\alpha \in \Delta_+$ and let Vir_α be the Virasoro algebra generated by the component operators of $Y(\omega_\alpha, z)$. Then one can compute that

$$L_\alpha(n) = \frac{1}{2k_\alpha(k_\alpha + 2)} \sum_{s+t=n} (-\circ h_\alpha(s)h_\alpha(t)\circ + k_\alpha \circ x_\alpha(s)x_{-\alpha}(t)\circ + k_\alpha \circ x_{-\alpha}(s)x_\alpha(t)\circ)$$

for $n \in \mathbb{Z}$, where

$$\circ u(s)v(t)\circ = \begin{cases} u(s)v(t) & \text{if } s < 0 \\ v(t)u(s) & \text{if } s \geq 0 \end{cases}$$

for $u, v \in \mathfrak{g}$ and $s, t \in \mathbb{Z}$.

Lemma 3.3. $\mathcal{L}(k, 0)$ is a unitary representation for the Virasoro algebra Vir_α for all $\alpha \in \Delta_+$. That is,

$$(L_\alpha(m)u, v) = (u, L_\alpha(-m)v)$$

for all $m \in \mathbb{Z}$. In particular, $L_\alpha(0)$ is a hermitian operator on $\mathcal{L}(k, 0)_n$ for all $n \in \mathbb{Z}$, and $K(\mathfrak{g}, k)$ is a unitary representation of Vir_α for all $\alpha \in \Delta_+$.

Proof. Let $u, v \in \mathcal{L}(k, 0)$ and $n \in \mathbb{Z}$. Then

$$\begin{aligned} & 2k_\alpha(k_\alpha + 2)(L_\alpha(n)u, v) \\ &= \sum_{s+t=n, s<0} (-h_\alpha(s)h_\alpha(t)u + k_\alpha x_\alpha(s)x_{-\alpha}(t)u + k_\alpha x_{-\alpha}(s)x_\alpha(t)u, v) \\ &+ \sum_{s+t=n, s\geq 0} (-h_\alpha(t)h_\alpha(s)u + k_\alpha x_{-\alpha}(t)x_\alpha(s)u + k_\alpha x_\alpha(t)x_{-\alpha}(s)u, v) \\ &= \sum_{s+t=n, s<0} (u, -h_\alpha(-t)h_\alpha(-s)v + k_\alpha x_{-\alpha}(-t)x_\alpha(-s)v + k_\alpha x_\alpha(-t)x_{-\alpha}(-s)v) \\ &+ \sum_{s+t=n, s\geq 0} (u, -h_\alpha(-s)h_\alpha(-t)v + k_\alpha x_\alpha(-s)x_{-\alpha}(-t)v + k_\alpha x_{-\alpha}(-s)x_\alpha(-t)v) \\ &= \sum_{s+t=n, s\leq 0} (u, -h_\alpha(-t)h_\alpha(-s)v + k_\alpha x_{-\alpha}(-t)x_\alpha(-s)v + k_\alpha x_\alpha(-t)x_{-\alpha}(-s)v) \\ &+ \sum_{s+t=n, s>0} (u, -h_\alpha(-s)h_\alpha(-t)v + k_\alpha x_\alpha(-s)x_{-\alpha}(-t)v + k_\alpha x_{-\alpha}(-s)x_\alpha(-t)v) \\ &= \sum_{s+t=-n} (u, -{}^\circ h_\alpha(s)h_\alpha(t){}^\circ v + k_\alpha {}^\circ x_\alpha(s)x_{-\alpha}(t){}^\circ v + k_\alpha {}^\circ x_{-\alpha}(s)x_\alpha(t){}^\circ v) \\ &= 2k_\alpha(k_\alpha + 2)(u, L_\alpha(-n)v), \end{aligned}$$

as desired. \square

We are now proving Theorem 3.1.

Proof. For short, we set $V = K(\mathfrak{g}, k)$. It follows from [26] that if $\mathcal{D}(V) = V$, then V is C_2 -cofinite. Assume $\mathcal{D}(V) \neq V$. Then there exists $v = (v^0, \dots, v^n, \dots) \in \mathcal{D}(V)$ not in V , where $v^i \in V_i = K(\mathfrak{g}, k)_i$. That is, there are infinitely many nonzero v^i . Recall that $k_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} k$. So $k = \frac{\langle \alpha, \alpha \rangle}{2} k_\alpha$. Assume that $k_\alpha \leq q$ for all $\alpha \in \Delta_+$, then P_α is regular for all $\alpha \in \Delta_+$ from the assumption. Thus the weak P_α -module $\mathcal{D}(V)$ is a direct sum of irreducible ordinary P_α -modules.

It is easy to see that if \mathfrak{g} is ADE type, then $k \leq q$, if \mathfrak{g} is type G_2 , then $k \leq [q/3]$, and if \mathfrak{g} is the other type, $k \leq [q/2]$. Since $\mathcal{D}(V)$ is a direct sum of irreducible ordinary P_α -modules, each $L_\alpha(0)$ is semisimple on $\mathcal{D}(V)$. This implies that v is a sum of eigenvectors for $L_\alpha(0)$ with rational eigenvalues $\lambda_1^\alpha, \dots, \lambda_{j_\alpha}^\alpha$ [9]. Since $L_\alpha(0)$ preserves each V_i for all i , we see that each v^i is a sum of eigenvectors for $L_\alpha(0)$ with possible eigenvalues $\lambda_1^\alpha, \dots, \lambda_{j_\alpha}^\alpha$. Let λ_α be the maximum of λ_j^α for $1 \leq j \leq j_\alpha$.

Recall that $L(0) = \sum_{\alpha \in \Delta_+} c_\alpha L_\alpha(0)$, where $c_\alpha = \frac{k(k_\alpha+2)}{k_\alpha(k+h^\vee)}$ is positive. It is clear that $(L(0)v^i, v^i) = i(v^i, v^i)$. By Lemma 3.3, each $L_\alpha(0)$ is a hermitian operator on V_i and

eigenvectors with different eigenvalues are orthogonal with respect to the positive definite hermitian form (\cdot, \cdot) . As a result, $(L_\alpha(0)v^i, v^i) \leq \lambda_\alpha(v^i, v^i)$ for all i . This yields

$$i(v^i, v^i) = (L(0)v^i, v^i) = \sum_{\alpha \in \Delta_+} c_\alpha (L_\alpha(0)v^i, v^i) \leq \sum_{\alpha \in \Delta_+} c_\alpha \lambda_\alpha(v^i, v^i).$$

for all i . Since there are infinitely many nonzero v^i , we conclude that there are infinitely many positive integers i less than or equal to a fixed number $\sum_{\alpha \in \Delta_+} c_\alpha \lambda_\alpha$. This is obviously a contradiction. The proof is complete. \square

4 The structure of $K(\mathfrak{g}, 1)$

In this section, we will discuss the parafermion vertex operator algebras for $k = 1$. We need the following Lemma from [4].

Lemma 4.1. *The parafermion vertex operator algebras $K(sl_2, 1) = \mathbb{C}$ and $K(sl_2, 2)$ is isomorphic to $L(\frac{1}{2}, 0)$. In particular, $W_\alpha^3 = 0$ in both cases, where α is a root of sl_2 .*

From Lemma 4.1, if $k = 1$ and α is a long root, then $\omega_\alpha = W_\alpha^3 = 0$. This shows that $K(\mathfrak{g}, 1) = \mathbb{C}$ if \mathfrak{g} is of ADE types. We will restrict ourselves to the non-simply laced simple Lie algebra in the rest of this section. Recall that $\langle \theta, \theta \rangle = 2$. We note that the subalgebra P_α of $K(\mathfrak{g}, 1)$ generated by $\omega_\alpha, W_\alpha^3$ with $\alpha \in \Delta_+$ is isomorphic to $K(sl_2, 2)$ if $\langle \alpha, \alpha \rangle = 1$ and isomorphic to $K(sl_2, 3)$ if $\langle \alpha, \alpha \rangle = \frac{2}{3}$. Furthermore, $K(sl_2, 2)$ is isomorphic to the rational vertex operator algebra $L(\frac{1}{2}, 0)$ which is the irreducible highest weight module for the Virasoro algebra with central charge $\frac{1}{2}$. So in the case that \mathfrak{g} is of type B_l, C_l, F_4 , $K(\mathfrak{g}, 1)$ is generated by ω_α with $\alpha \in \Delta_+$ being short roots. If \mathfrak{g} is of type G_2 , the situation is more complicated. The result is as follows: the parafermion vertex operator algebra associated to the non-simply laced simple Lie algebra with $k = 1$ is the same as the parafermion vertex operator algebra obtained from the simply laced simple Lie algebra whose Dynkin diagram is obtained from the Dynkin diagram of the non-simply laced algebra by deleting the long roots with $k = 3$ in the case of G_2 and with $k = 2$ in the rest of cases. This result has been observed in [16] by using the partition functions. Recall that the central charge of the parafermion vertex operator algebra is given by $\frac{klh-lh^\vee}{k+h^\vee}$.

In the following, we fix an orthonormal basis $\{\epsilon_1, \dots, \epsilon_l\}$ of Euclidean space \mathbb{R}^n and we refer the reader to [18] for the details on root systems. We denote the set of short roots by Δ^s and the set of positive short roots by Δ_+^s .

Recall that Q is the root lattice of \mathfrak{g} . Denote by Q_L the sublattice generated by the long roots. The following lemma is important in our discussion below.

Lemma 4.2. *Assume that $\alpha, \beta \in \Delta^s$ such that $\alpha \in Q_L \pm \beta$, then $\omega_\alpha = \omega_\beta$, $W_\alpha^3 = \pm W_\beta^3$ in $K(\mathfrak{g}, 1)$.*

Proof. The result is obvious if $\alpha = \pm\beta$. We only need to deal with the case that $\alpha \neq \pm\beta$. Suppose that $\alpha \in Q_L + \beta$.

We first prove that $\gamma = \alpha - \beta$ is a long root. Note that $\langle \gamma, \gamma \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle - 2\langle \alpha, \beta \rangle$ is a positive even integer as Q_L is a positive definite even lattice. If \mathfrak{g} is of type of G_2 ,

then $\langle \gamma, \gamma \rangle = \frac{4}{3} - 2\langle \alpha, \beta \rangle$ is an even integer greater than or equal to 2. This forces $\langle \alpha, \beta \rangle$ to be $-\frac{1}{3}$ and γ is a long root. If \mathfrak{g} is not of type G_2 , $\langle \gamma, \gamma \rangle = 2 - 2\langle \alpha, \beta \rangle$. The possible values for $\langle \alpha, \beta \rangle$ is 0 or -1 . If $\langle \alpha, \beta \rangle = -1$, then $\alpha = -\beta$ and this is impossible from the assumption. So $\langle \alpha, \beta \rangle = 0$ and $\langle \gamma, \gamma \rangle = 2$. Again γ is a root.

Let σ_γ be the reflection associated to root γ . It is easy to verify that $\sigma_\gamma(\alpha) = \beta$. Let $\tau_\gamma = e^{x_{-\gamma}(0)}e^{-x_\gamma(0)}e^{x_{-\gamma}(0)}$. Then τ_γ is an automorphism of $\mathcal{L}(1, 0)$. Since τ_γ preserves $h(-1)\mathbb{1}$ for $h \in \mathfrak{h}$, we see that τ_γ preserves $K(\mathfrak{g}, 1)$. In fact, $\tau_\gamma(\omega_\alpha) = \omega_\beta$, $\tau_\gamma(W_\alpha^3) = W_\beta^3$. So it is sufficient to show that $\tau_\gamma = 1$ on $K(\mathfrak{g}, 1)$.

Let Δ^l be a subset of Δ consisting of long roots. Then Δ^l is a root lattice which is orthogonal union of irreducible root systems of ADE types. Consider the vertex operator subalgebra U of $\mathcal{L}(1, 0)$ generated by $x_\phi(-1)\mathbb{1}$ for $\phi \in \Delta^l$. Then $U = V_{Q_L}$ is a lattice vertex operator algebra. It is well known that the Virasoro element of U is given by

$$\omega_U = \frac{1}{2} \sum_{i=1}^s u^i(-1)^2 \mathbb{1}$$

where $\{u^1, \dots, u^s\}$ is an orthonormal basis of $\sum_{\phi \in \Delta^l} \mathbb{C}h_\phi$ with respect to $\langle \cdot \rangle$. Since

$$u^i(n)K(\mathfrak{g}, 1) = 0$$

for $i = 1, \dots, s$ and $n \geq 0$, we see that $L_U(-1)K(\mathfrak{g}, 1) = 0$, where $L_U(-1)$ is the component operator of $Y(\omega_U, z) = \sum_{m \in \mathbb{Z}} L_U(m)z^{-m-2}$. This implies that $u_m K(\mathfrak{g}, 1) = 0$ for all $u \in U$ and $m \geq 0$. In particular, $x_\gamma(0) = x_{-\gamma}(0) = 0$ on $K(\mathfrak{g}, 1)$. As a result, $\tau_\gamma = 1$ on $K(\mathfrak{g}, 1)$. So we have proved that if $\alpha + Q_L = \beta + Q_L$, then $\omega_\alpha = \omega_\beta$, $W_\alpha^3 = W_\beta^3$.

Similarly, if $\alpha \in Q_L - \beta$, then $\omega_\alpha = \omega_{-\beta} = \omega_\beta$, $W_\alpha^3 = W_{-\beta}^3 = -W_\beta^3$. The proof is complete. \square

We also need to generalize (3) of Theorem 2.1 to an arbitrary subalgebra of \mathfrak{g} .

Lemma 4.3. *Let \mathfrak{g}_1 be simple subalgebra of \mathfrak{g} with root system $\Delta_1 \subset \Delta$. Then the vertex operator subalgebra of $K(\mathfrak{g}, k)$ generated by $\omega_\alpha, W_\alpha^3$ for $\alpha \in \Delta_1$ is isomorphic to $K(\mathfrak{g}_1, k_1)$, where k_1 is defined as follows: $k_1 = k$ if $\Delta_1 \not\subset \Delta^s$; $k_1 = 3k$ if \mathfrak{g} is of type G_2 and $\Delta_1 \subset \Delta^s$; $k_1 = 2k$ if \mathfrak{g} is of type B_l, C_l, F_4 and $\Delta_1 \subset \Delta^s$.*

Proof. The proof is similar to that of Proposition 4.6 of [11] by noting that for each $\alpha \in \Delta$, $\mathcal{L}(k, 0)$ is an integrable module for the affine Lie algebra $\widehat{\mathfrak{g}}_\alpha$. \square

We are now ready to discuss the parafermion vertex operator algebra $K(\mathfrak{g}, 1)$ if \mathfrak{g} is a non-simply laced simple Lie algebra. We denote the root system whose simple roots are the short simple roots of the non-simply laced Lie algebra by Δ' .

4.1 B_l

The root system is given by

$$\Delta = \{\pm \epsilon_i, \pm(\epsilon_i \pm \epsilon_j) | i, j = 1, \dots, l, i \neq j\}$$

with simple roots

$$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\}$$

and $\Delta' = \{\pm\epsilon_l\}$ is the root system of type A_1 .

In this case, $K(B_l, 1)$ is generated by ω_{ϵ_i} for $i = 1, \dots, l$ and each ω_{ϵ_i} generates a vertex operator algebra isomorphic to $L(\frac{1}{2}, 0)$. Note that the central charge of $K(B_l, 1)$ is equal to $\frac{1}{2}$. So $K(B_l, 1)$ is an extension of $L(\frac{1}{2}, 0)$. But the only extension of $L(\frac{1}{2}, 0)$ is itself due to the integral weight restriction. As a result, $K(B_l, 1) = L(\frac{1}{2}, 0)$ and $\omega_{\epsilon_i} = \omega$ for all i . One can also use Lemma 4.2 to see that $\omega_{\epsilon_i} = \omega_{\epsilon_j}$ for all i, j .

4.2 C_l

Assume that $l \geq 3$. The root system is given by

$$\Delta = \{\pm 2\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) | i, j = 1, \dots, l, i \neq j\}$$

with simple roots

$$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{l-1} - \epsilon_l, 2\epsilon_l\}$$

and

$$\Delta' = \{\epsilon_i - \epsilon_j | i \neq j\}$$

is a root system of type A_{l-1} . By Lemma 4.2, we see that $\omega_{\epsilon_i - \epsilon_j} = \omega_{\epsilon_i + \epsilon_j}$ if $i \neq j$. Thus $K(C_l, 1)$ is generated by $\omega_{\epsilon_i - \epsilon_j}$ for $i < j$.

Note that

$$\mathfrak{g}_1 = \sum_{i \neq j} (\mathbb{C}x_{\epsilon_i - \epsilon_j} + \mathbb{C}h_{\epsilon_i - \epsilon_j})$$

is a subalgebra of C_l isomorphic to A_{l-1} . As a result, the vertex operator algebra $K(C_l, 1)$ is isomorphic to $K(A_{l-1}, 2)$ by Lemma 4.3.

4.3 F_4

The root system is given by

$$\Delta = \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j), \pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) | i, j = 1, \dots, 4, i \neq j\}$$

with simple roots

$$\{\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4, \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$$

and

$$\Delta' = \{\pm\epsilon_4, \pm\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 \pm \epsilon_4)\}$$

is a root system of type A_2 . By Lemma 4.3 again, $K(F_4, 1)$ is isomorphic to $K(A_2, 2)$.

We now determine $K(A_2, 2)$ explicitly. Let $\{\alpha_1, \alpha_2\}$ be the simple roots of Δ . Set $\omega' = \omega - \omega_{\alpha_1}$. Then ω' is a Virasoro vector with central charge $\frac{7}{10}$. By Lemma 3.3,

$K(A_2, 2)$ is a unitary representations for both Vir_{α_1} and the Virasoro algebra generated by the component operators of $Y(\omega', z)$. Let U be a vertex operator subalgebra of $K(A_2, 2)$ generated by ω_{α_1} and ω' . Then U is isomorphic to rational vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0)$. So $K(A_2, 2)$ is a completely reducible U -module. Note that the irreducible $L(\frac{7}{10}, 0)$ -modules are $L(\frac{7}{10}, h)$ with

$$h = 0, \frac{7}{16}, \frac{3}{80}, \frac{3}{2}, \frac{3}{5}, \frac{1}{10}$$

[27] and $L(\frac{1}{2}, 0)$ is a rational vertex operator algebra with 3 irreducible modules $L(\frac{1}{2}, h')$ with $h' = 0, \frac{1}{2}, \frac{1}{16}$ [10], [27]. Since $K(A_2, 2)$ only has integral weight, we see that as a U -module, $K(A_2, 2)$ can only be U or $U \oplus n(L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}))$. But $K(A_2, 2)_2$ is 3-dimensional with basis $\omega_{\alpha_1}, \omega_{\alpha_2}, \omega_{\alpha_1+\alpha_2}$, thus we conclude that

$$K(A_2, 2) = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}).$$

4.4 G_2

The root system is given by

$$\Delta = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_2 - \epsilon_3), \pm(\epsilon_1 - \epsilon_3), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3), \pm(2\epsilon_2 - \epsilon_1 - \epsilon_3), \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\}$$

with simple roots

$$\{\epsilon_1 - \epsilon_2, -2\epsilon_1 + \epsilon_2 + \epsilon_3\}$$

and $\Delta' = \{\pm(\epsilon_1 - \epsilon_2)\}$ is a root system of type A_1 . By Lemma 4.2, we deduce that $K(G_2, 1)$ is generated by $\omega_{\epsilon_1 - \epsilon_2}$ and $W_{\epsilon_1 - \epsilon_2}^3$. Thus by Lemma 4.3, $K(G_2, 1)$ is isomorphic to $K(A_1, 3)$ which is isomorphic to $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ [4]. The vertex operator algebra $K(A_1, 3)$ is rational and the irreducible modules has been classified in [19].

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